Counting with weights

Problem 4. (Iran 1999) Suppose that $C_1, \dots, C_n$ ($n \geq 2$) are circles of radius one in the plane such that no two of them are tangent, and the subset of the plane formed by the union of these circles is connected. Let S be the set of points that belong to at least two circles. Show that $|S| \geq n$.

Let us set up a matrix with n columns, each representing an unit circle, and $|S|$ rows, each representing an intersection point. An entry is 1 if the corresponding point lies on the corresponding circle and 0 otherwise. Since no circle is disjoint from the rest, every column contains at least two 1's as no two circles are tangent. As well, by definition, each row must contain at least two 1's.

Let one focus one 1 in the incidence matrix, say $a_{ij} = 1$. Each one on row i distinct from $a_{i,j}$ corresponds to a circle that goes through the point represented by row i. Any such circle meets the circle C_j at exactly two points as no tangency is allowed. So we will associate each one in row i distinct from $a_{i,j}$ with a one from column j different from $a_{i,j}$ that represents the second intersection. Note that no one in column j is associated with two different 1's on row i, as this would mean that three different unit circles are passing through the same two points, which is impossible. Hence, there is an injection from the 1's in row i to the 1's in column j.

$$\left(\begin{array}{ccccccc}
 & & & \vdots & & & \\
 & 1 & \rightarrow & 1 & & & \\
 & \uparrow & & \vdots & & & \\
\cdots & 1 & \cdots & a_{i,j} = 1 & \cdots & 1 & \cdots \\
 & & & \vdots & & \downarrow & \\
 & & & 1 & \leftarrow & 1 & \\
 & & & \vdots & & &
\end{array}\right)$$

How can we use this information? Well, this is where the weights come in.

Let us revisit the idea of counting 1's. However, this time, we will assign a "weight" to each 1. For example, if an incidence matrix has three 1's on each row, and we assign a weight of $\frac{1}{3}$ to each 1, then the sum of all the weights is r, the number of rows. We will see momentarily why this might be useful.

Using the same idea, if we associate each one with a weight, in such a way the weights of all the 1's in each row sum to 1, then the sum of the weights of all the 1's in the matrix equals to r. The following proposition comes from this idea.

Proposition 3. Let $A = (a_{i,j})$ be an $r \times c$ matrix with row sums R_i, and column sums C_j. If $R_i > 0$ for $1 \leq i \leq r$, then

$$\sum_{i,j} \frac{a_{i,j}}{R_i} = r.$$

Similarly, if $C_j > 0$ for $1 \leq j \leq c$, then

$$\sum_{i,j} \frac{a_{i,j}}{C_j} = c.$$

Proof. We have

$$\sum_{i,j} \frac{a_{i,j}}{R_i} = \sum_{i=1}^{r} \left(\frac{1}{R_i} \sum_{j=1}^{c} a_{i,j} \right) = \sum_{i=1}^{r} \left(\frac{1}{R_i} \ R_i \right) = \sum_{i=1}^{r} 1 = r.$$

The proof of the second part is done in a similar manner. $\square$

The following proposition leads to an application of this idea.

Proposition 4. Let $A = (a_{i,j})$ be a $r \times c$ $(0,1)$-matrix with row sums R_i and column sums C_j such that $R_i > 0$ and $C_j > 0$ for $1 \le i \le r$ and $1 \le j \le c$. If $C_j \ge R_i$ whenever $a_{i,j} = 1$, then $r \ge c$.

Proof. When $a_{i,j} = 1$, $R_i \le C_j$ implies that $\frac{1}{R_i} \ge \frac{1}{C_j}$. It follows that $\frac{a_{i,j}}{R_i} \ge \frac{a_{i,j}}{C_j}$ for all $1 \le i \le r$ and $1 \le j \le c$. From Proposition 3, we have

$$r = \sum_{i,j} \frac{a_{i,j}}{R_i} \ge \sum_{i,j} \frac{a_{i,j}}{C_j} = c. \qquad \square$$

Back to the problem. Since there is an injective mapping from the 1's in row i to the 1's in column j, we see that $C_j \ge R_i$ whenever $a_{i,j} = 1$. Therefore, therefore $r \ge c$, and the result follows. $\square$.

We will play one more variation on this technique. Sometimes we may not be able to compare R_i and C_j when $a_{i,j} = 1$, but we may be able to make the comparison when $a_{i,j} = 0$. The next proposition is an analogue of Proposition 4.

Proposition 5. Let $A = (a_{i,j})$ be an $r \times c$ $(0,1)$-matrix with row sums R_i, and column sums C_j. If $0 < R_i < c$ and $0 < C_j < r$ for $1 \le i \le r$ and $1 \le j \le c$, and $C_j \ge R_i$ whenever $a_{i,j} = 0$, then $r \ge c$.

Proof. Suppose on the contrary that $r < c$. Then $0 < r - C_j < c - R_i$ whenever $a_{i,j} = 0$. Hence, $\frac{1}{c-R_i} < \frac{1}{r-C_j}$, which implies

$$\frac{R_i}{c - R_i} < \frac{C_j}{r - C_j}.$$

Let M denote the number of 1's in A, we have

$$\begin{aligned} M = \sum_{i=1}^{r} R_i = \sum_{i=1}^{r} (c - R_i) \frac{R_i}{c - R_i} &= \sum_{i=1}^{r} \left(\sum_{i=1}^{c} (1 - a_{i,j}) \right) \frac{R_i}{c - R_i} = \sum_{i,j} \frac{(1 - a_{i,j}) R_i}{c - R_i} \\ < \sum_{i,j} \frac{(1 - a_{i,j}) C_i}{r - C_j} &= \sum_{j=1}^{c} \left(\sum_{i=1}^{r} (1 - a_{i,j}) \right) \frac{C_j}{r - C_j} = \sum_{j=1}^{c} (r - C_j) \frac{C_j}{r - C_j} = M \end{aligned}$$

This is clearly impossible. Therefore, $r \ge c$. $\square$

Problem 5. Let $S_1, S_2, \ldots, S_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ such that $|S_i \cap S_j| = 1$ for all $i \ne j$. Prove that $m \le n$.

This problem is a special case of Fisher's inequality. It has a very simple and element proof using linear algebra. However, in this note, we give the combinatorial solution to the problem following the models that we have developed so far.

Proof. The result holds trivially if the collection is empty ($m = 0$) or $m = 1$. So we may assume that $m \ge 2$. It is easy to see that none of the sets S_i are empty. So assume $m \ge 2$ and all of the sets are non-empty.

As usual, we consider the incidence matrix A for the collection of sets. The m rows of A correspond to sets and the n columns correspond to the elements, where $a_{i,j}$ is 1 if element j belongs to set S_i, and is 0 otherwise.

Now let us show that the hypotheses of Proposition 5 are satisfied. If any row has all 1's, say the first row, then the constraint $|S_1 \cap S_i| = 1$ for all $i \ne 1$ forces $|S_i| = 1$, which, along with $|S_i \cap S_j| = 1$, implies that $m = 2$, and $n \ge 2$ because the sets are distinct. If any column has all zeros, then that element belongs to none of the sets and we may simply remove that column. We may do this until every column satisfies $C_j \ge 1$ because if the result holds for this reduced matrix, it certainly holds for the original A. Finally, if any column has all 1's, say the first column, then $|S_i \cap S_j| = 1$ implies that no other column may contain two 1's. As well, at most one row may contain a single one (on the first column), and each of the other $r - 1$ rows must have the second one on distinct columns. So the number of columns must be greater than or equal to the number of rows, giving $m \le n$ in this case as well. We are now ready to employ Proposition 5.

$$\begin{pmatrix}
 & & & \vdots & & & \\
 & & & 1 & \rightarrow & 1 & \\
 & & & \vdots & & \downarrow & \\
\cdots & 1 & \cdots & a_{i,j}=0 & \cdots & 1 & \cdots \\
 & \uparrow & & \vdots & & & \\
 & 1 & \leftarrow & 1 & & & \\
 & & & \vdots & & &
\end{pmatrix}$$

Let us consider any $a_{i,j} = 0$. By the given condition, for every one on column j, its corresponding subset must intersect with A_i. So we may correspond each one on C_j with an one on row i such that the element represented by the one on R_i also belongs to the subset represented by the one on C_j. Note that this correspondence is injective, since having two 1's on C_j both corresponding to the same one in R_i implies that some two subsets intersect in at least two elements. The injective mapping implies that there must be at least as many 1's on i^{th} row as there are on the j^{th} column.

Thus $R_i \geq C_j$ for any $a_{i,j} = 0$. It follows from Proposition 5 (with the roles of rows and columns interchanged) that $m \leq n$. □

Final remarks Incidence matrices can be very useful for visualizing the combinatorial configuration. However, when writing up a solution, it's usually easier to avoid the incidence matrix and simply stick with set theory notation instead.